Critical "dimension" in shell model turbulence

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We investigate the Gledzer-Ohkitani-Yamada (GOY) shell model within the scenario of a critical dimension in fully developed turbulence. By changing the conserved quantities, one can continuously vary an "effective dimension" between $d=2$ and $d=3$. We identify a critical point between these two situations where the flux of energy changes sign and the helicity flux diverges. Close to the critical point the energy spectrum exhibits a turbulent scaling regime followed by a plateau of thermal equilibrium. The corrections due to intermittency persist close to the critical point. We identify scaling laws and perform a rescaling argument to derive a relation between the critical exponents. We further discuss the distribution function of the energy flux.

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Many theoretical and experimental results for fully developed turbulence have been offered over the last decade. A different approach has been presented by Yakhot [1] in which the method of generating functions by Polyakov [2] is generalized to the Navier-Stokes equations. Applying a renormalization group procedure [3] results in an estimate of a critical dimension for turbulence, around $d_c \sim 2.5$, thus following the foot steps of an original idea by Frisch and Fournier [4] but correcting the actual value of the dimension. The physical idea behind the existence of a critical dimension is related to the well known fact that the energy cascade in three dimensional turbulence is "forward" (in k space) going from large to small scales, whereas for two dimensional turbulence it is backward, from small to large scales. This leads to the identification of a critical dimension between two and three at which the flux of energy changes its sign, and the amplitude of the field turns into a peak where there is no flux neither forward nor backward. In Ref. [1] the theory is expanded around this critical point in terms of a ratio between two time scales. However, it is not possible to investigate the physical behavior in a noninteger dimension directly, neither experimentally nor numerically. In this paper we therefore propose to study this type of criticality in a shell model for turbulence [5]. In particular we focus on the Gledzer-Ohkitani-Yamada (GOY) model [6–8] which exhibits well known conservation laws: in the three dimension (3D) version energy and helicity are conserved; in the two dimension (2D) version energy and enstrophy are conserved. It is possible to *continuously* vary the "effective dimension" of the model by changing the second conserved quantity from a helicity to an enstrophy quantity. As the energy is *always* conserved, we can study the energy flux directly as a function of the variation in the second conserved quantity and we identify a critical point, where the flux changes sign. Indeed the second conserved quantity is nonphysical at this point as expected. Nevertheless we are able numerically to extract a series of different properties of the spectrum and the probabilty density function (PDF) around this critical point. A similar observation of a change of sign in the energy flux as a function of a parameter was already made in a different shell model by Bell and Nelkin [9]. In their model the dynamics is not intermittent and the properties of the model are thus quite different from the GOY model.

Our starting point is the approximative approach to turbulence made by discretizing the wave number space by exponentially separating "shells" [5]. In this respect, we apply the GOY model [6,7] which has been successful in giving results for intermittency corrections in agreement with experiments [8]; an agreement that is observed despite the fact that experiments usually monitor spatial intermittency, whereas the GOY model exhibits temporal intermittency (for other results on the GOY model, see [10–12]). The starting point is a set of wave numbers $k_n = k_0 r^n$ ($r=2$ and $k_0=1$ in this paper) and an associated complex amplitude u_n of the velocity field. Each amplitude interacts with nearest and next-nearest neighboring shells and the corresponding set of coupled ordinary differential equations takes the form

$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n = ik_n\left(a_n u_{n+1}^* u_{n+2}^* + \frac{b_n}{2} u_{n-1}^* u_{n+1}^* + \frac{c_n}{4} u_{n-1}^* u_{n-2}^*\right) + f_{n_f}, \tag{1}$$

with $n=1,\ldots N$, and boundary conditions $b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0$. The values of the coupling constants are fixed by imposing conserved quantities. By conserving the total energy $\Sigma_n |u_n|^2$ when $f_{n_f} = \nu = 0$, we obtain the constraints $a_n + b_{n+1} + c_{n+2} = 0$. The time scale is fixed by the condition $a_n = 1$ leaving free the parameter δ by defining the coupling constants as

$$a_n = 1, \quad b_n = -\delta, \quad c_n = -(1-\delta). \tag{2}$$

The model also possesses a second conserved quantity of the form

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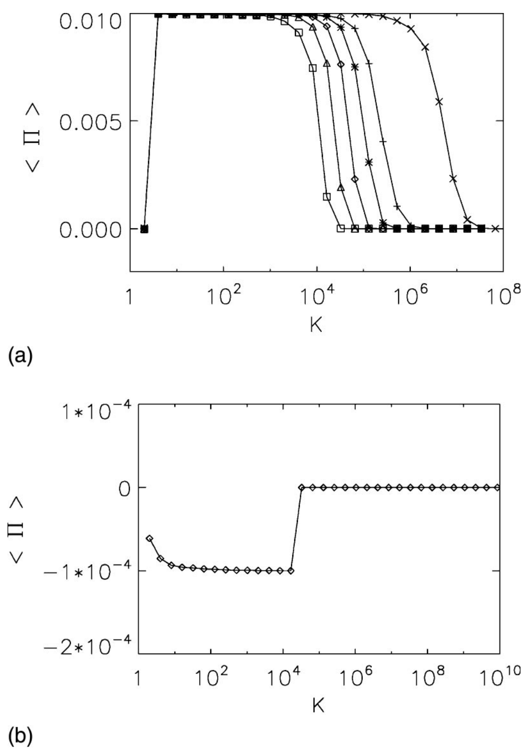


FIG. 1. (a) Average energy flux vs the wave number k with $N=25$, $\nu=10^{-10}$, $n_f=2$ for $\delta=0.5(\times)$, and $\epsilon=2\times10^{-4}(+)$, 2×10^{-5} $(*)$, . . . ,2×10^{-8} ($\square$). Note that as ϵ decreases the inertial range shrinks. (b) Inverse energy flux with $N=33$, $\nu=10^{-16}$ for $\epsilon=-0.125$. The forcing is on shell 15, $f_{15}=(1+i)*10^{-4}/u_{15}^{*}$. A large scale viscosity is now applied (see the text).

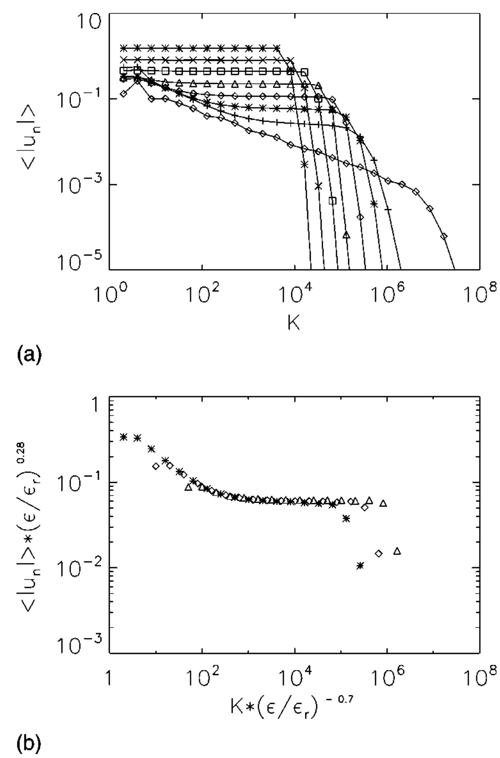


FIG. 2. (a) The spectrum $\langle|u|\rangle$ versus k for the 3D case $\delta=0.5$ ($\diamond$) and for $\epsilon=2\times10^{-4}$ $(+)$, . . . ,$\epsilon=2\times10^{-10}$ $(*)$. Note the flat part of the spectrum developing as $\epsilon\to0$. (b) A rescaling plot of the spectra in Fig. 2 by $\langle|u|\rangle*(\epsilon/\epsilon_r)^{0.28}$ vs $k*(\epsilon/\epsilon_r)^{-0.7}$ [$\epsilon=10^{-6}$ ($\diamond$) and $\epsilon=10^{-7}$ ($\triangle$)], where $\epsilon_r=2\times10^{-5}$ $(*)$.

$$Q=\sum k_n^{\alpha}|u_n|^2, \tag{3}$$

which leads to a relation between α and δ: $2^{\alpha}=1/(\delta-1)$. For $\delta<1$ this relation requires complex values of α, with $\mathrm{Im}(\alpha)=\pi/\ln 2$. In 3D turbulence helicity $H=\int(\nabla\times\mathbf{u}(\mathbf{x}))\cdot\mathbf{u}(\mathbf{x})d\mathbf{x}$ is conserved, which in terms of shell variables takes the form [11]

$$H=\sum_n (-1)^n k_n|u_n|^2, \tag{4}$$

when the values of parameters are $\delta=\frac{1}{2}$ and $\mathrm{Re}(\alpha)=1$. In 2D turbulence on the other hand enstrophy $\Omega=\int|\nabla\times\mathbf{u}(\mathbf{x})|^2d\mathbf{x}$ is conserved and this corresponds to the parameters $\delta=\frac{5}{4}$, $\alpha=2$ which on the shells takes the form $\Omega=\Sigma k_n^2|u_n|^2$. Note that energy is conserved for any value of δ. This gives us the possibility to *continuously* vary the effective dimension [and thus the generalized helicity/enstrophy (3)] by varying the parameter δ between $\delta=\frac{1}{2}$ (3D) and $\delta=\frac{5}{4}$ (2D).

The critical point is identified by looking at the energy flux through each shell which is given by [10]

$$\Pi_n=\left\langle -\frac{d}{dt}\sum_{i=1}^{n}\left|u_i\right|^2\right\rangle$$
$$=\left\langle -\mathrm{Im}\left(k_n u_n u_{n+1}\left(u_{n+2}+\frac{(1-\delta)}{2}u_{n-1}\right)\right)\right\rangle, \tag{5}$$

where only the contributions of the nonlinear terms are considered in the time rate of change of the cumulative energy. From Eq. (5) we see that the last term vanishes as $\delta\to1$, causing a depletion in the energy transfer [12]. This is observed in the numerical simulations shown in Fig. 1(a), where the inertial range of the flux shrinks as $\epsilon\equiv1-\delta\to0$. Note that in Fig. 1(a) (as well as in all the subsequent simulations concerning the 3D side for $\delta<1$) we apply a forcing on the form $f_{n_f}=(1+i)\times10^{-2}/u_{n_f}^{*}$, with $n_f=2$, in order to ensure a constant input of the energy (and thus a constant flux). We reach similar conclusions when a constant deterministic force is applied. Moving above $\delta=1$, the energy flux reverses going instead from small to large scales, see Fig. 1(b). Therefore the point $\delta=\delta_c=1$ defines a critical point where the energy flux for finite value of energy input f discontinuously jumps from positive to negative values (the jump diminishes with the forcing amplitude f). Furthermore, the rate of injected generalized helicity [13], given by $\Sigma_n(-1)^n k_n^{\alpha}\,\mathrm{Re}\langle f_n u_n^{*}\rangle$, diverges at this point [14].

Let us now turn to the spectra for $\delta<1$. As δ is increased from $\delta=0.5$ and $\epsilon\to0$, the regime in k space of the energy transfer diminishes. Because energy transport is more difficult it is likely that a part of the spectrum, for large values of k_n, is in thermal equilibrium. This is shown in Fig. 2, where the parameter ϵ is varied 1 decade for each spectrum (note that in our spectra we plot $\langle|u_n|\rangle$ vs k_n, which provides similar information as plotting $\langle|u_n|^2\rangle$). We identified various scaling laws associated with the spectra of Fig. 2. First of all the dissipative cutoff k_d moves in a systematic fashion as a function of ϵ. We find the following scaling law: $k_d \sim \epsilon^{\alpha_d}$, with $\alpha_d \simeq 0.3$. The plateau in the spectra (equipartition of energy among the shells) corresponds to the thermal equilibrium which overcomes the turbulent regime when the forward transfer of energy is reduced. We found that the level of the plateau scales with ϵ as $\langle|u_n|\rangle_{\rm pl} \sim \epsilon^{-\alpha_{\rm pl}}$, where $\alpha_{\rm pl} \simeq 0.28$, thus finally turning into a diverging amplitude around the forcing scale. The turbulent, cascading, part of the spectrum varies like $\langle|u_n|\rangle \sim k^{-\alpha_s}$, $\alpha_s \simeq 0.4$, taking into account corrections due to intermittency [8]. Finally, the critical wave number k_c, at which the spectrum crosses over from turbulent behavior to thermal equilibrium, also moves with ϵ, possibly like $k_c \sim \epsilon^{\alpha_c}$. To determine α_c, we balance the contributions from the two regimes

$$\langle|u_n|\rangle \sim \langle|u_n|\rangle_{\rm pl} \Rightarrow k^{-\alpha_s} \sim \epsilon^{-\alpha_{\rm pl}}, \tag{6}$$

and obtain the following scaling law for k_c:

$$k_c \sim \epsilon^{\alpha_c}, \alpha_c = \alpha_{\rm pl}/\alpha_s \simeq 0.7, \tag{7}$$

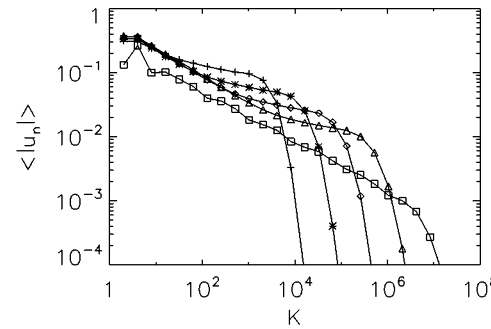
showing that the scaling exponents are not all independent [15]. This result can be verified by a simple rescaling of data. Let us assume that $\langle|u_n|\rangle/\langle|u_n|\rangle_{\rm pl}$ is a function of k/k_c alone, i.e.,

$$\frac{\langle|u_n|\rangle}{\langle|u_n|\rangle_{\rm pl}} \sim f\left(\frac{k}{k_c}\right), \tag{8}$$

where $f(x)$ is such that $f(x) \sim x^{-\alpha_s}$, $x \ll 1$, and $f(x) \sim \text{const}$, $x \gg 1$. Then a data collapse is obtained by plotting $\langle|u_n|\rangle/\epsilon^{-\alpha_{\rm pl}}$ versus k/ϵ^{α_c}. A good rescaling plot is obtained, see Fig. 2(b), when the estimated value $\alpha_c = 0.7$ is used. Notice that the collapse of data does not apply to the dissipative range, since k_d and k_c scale differently with ϵ. Also notice that the rescaling has been done by keeping a reference spectrum fixed (for $\epsilon = \epsilon_r$) and let the other spectra collapse onto it. Since there is no transfer of energy at the critical point, the nonlinear terms will not play any role and the equations will only include the dissipation and forcing terms. This can be made quantitative by the fact that the state with a peak at the forcing scale

$$\mathbf{u} = \left(0,0,0,\frac{f}{\nu k_{n_f}^2},0,0,\ldots,0\right) \tag{9}$$

is a fixed point of Eq. (1), which at $\epsilon = 0$ is marginally stable [16]. Indeed we find numerically at $\epsilon = 0$ that by starting with the fixed point Eq. (9) the peak stays at the forcing scale and the amplitudes remain zero above but become nonzero, although small, below the forcing scale. On the contrary, for $\epsilon \to 0^+$, the peak is unstable and the energy is soon redistributed to the neighboring shells. The existence of a sharp transition into the critical point was already indicated by a calculation of the maximal Lyapunov exponent which drops sharply as $\epsilon \to 0$ [16]. In order to study the behavior of the spectrum around the critical point in detail we have two parameters to vary, the "dimension" parameter δ and the viscosity ν. Figure 3(a) shows a series of spectra for $\epsilon = 0.002$ varying ν. Again one observes the shoulder at large k. The shoulder moves to higher k when the viscosity decreases. The wave vector for the dissipative cutoff k_d moves in the Kolmogorov fashion $k_d \sim \nu^{-3/4}$. The critical wave number k_c seems to scale with ν in the same way as k_d. This is confirmed by a finite size rescaling plot [see Fig. 3(b)], where, using scaling arguments similar to what were used for the ϵ variation, we rescale the k axis by $(\nu/\nu_{10})^{3/4}$ and the velocity axis by $(\nu/\nu_{10})^{-0.30}$.



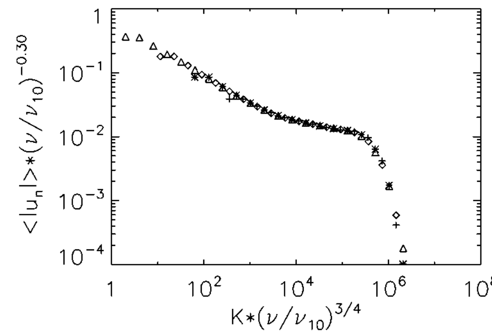


FIG. 3. (a) The spectrum $\langle|u|\rangle$ vs k for the standard 3D case $\delta = 0.5$, $\nu = 10^{-10}$ (□), and for $\epsilon = 2\times10^{-3}$ and $\nu = 10^{-10}$ (△), . . . ,10^{-7} (+). (b) A rescaling of the curves in (a) by $\langle|u|\rangle*(\nu/\nu_{10})^{-0.30}$ vs $k*(\nu/\nu_{10})^{3/4}$, where $\nu_{10} = 10^{-10}$.

Now consider the other side of the critical point, namely on the "two-dimensional" side for $\delta > 1$. The behavior of 2D shell models has been previously investigated in several papers [17–19]. A kind of "coupled GOY model" [18] gives an inverse flux of energy, which is explained in terms of a mean diffusive drift in a system close to statistical equilibrium, and shell models for 2D do not seem to give an inverse energy cascade with the usual "5/3" spectrum. In order to extract the energy of the inverse cascade we need to add a large scale viscosity to Eqs. (1) of the type $-\nu' k_n^{-2} u_n$. In Fig. (4) we show the energy spectrum for the case $\delta = 1.125$, $\nu = 10^{-16}$, and $\nu' = 10^{-1}$. The two branches of statistical equi-

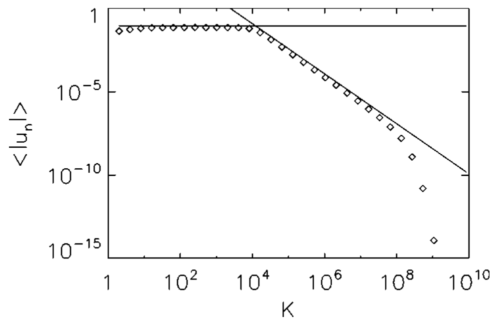


FIG. 4. $\langle |u| \rangle$ vs k for $\delta = 1.125$ $(\alpha = 3)$. The forcing is $f_{15} = (1+i) * 10^{-4}/u_{15}^*$. The two branches of statistical equilibrium $\langle |u| \rangle \sim$ const (energy equilibrium) and $\langle |u| \rangle \sim k^{-3/2}$ (generalized enstrophy equilibrium) are clearly visible.

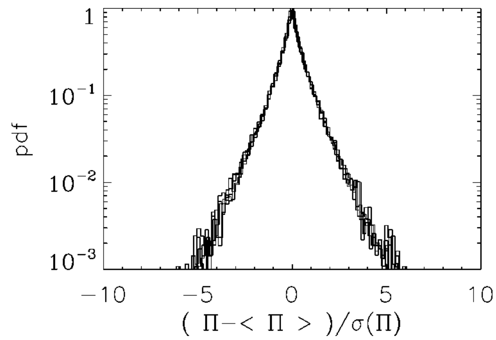


FIG. 5. Rescaling of the PDF of the energy flux obtained for shells in the "flat" part of the spectrum in Fig. 2 with $\nu = 10^{-10}$, $\epsilon = 3 \times 10^{-8}$, for the shells $n = 4,6,8,10,12$. The PDF's in each case are rescaled by the standard deviation of the energy flux.

librium, energy and generalized enstrophy equilibrium, respectively, are clearly visible (see [19] for details).

Let us turn our attention to the PDF for the energy flux at different scales. It is well known that, in fully developed turbulence, the PDF (of velocity increments) at the largest scales (small k) typically behaves like a Gaussian, slowly changing its form as one moves toward the small scales (large k), turning into a shape where large events play an important role giving a kind of stretched exponential PDF. Such nontrivial behavior is related to the property of multiscaling of the structure functions. When a turbulent scaling regime is detectable as in Fig. 3, we observe that the intermittency corrections appear to persist, even close to $\epsilon = 0$. However, in the flat part of the spectrum of Fig. 2, the probability distribution becomes wider at larger k but such that the shape is simply rescaled onto a universal and almost symmetric curve (rescaling by the standard deviation), see Fig. 5.

In this paper we have presented results for the existence of a critical point in the "GOY" shell model. Our main results can be summarized as follows. At this critical point, which lies between three- and two-dimensional behavior, the energy flux changes its sign, going from a forward to a backward transfer. Approaching this point from the "three-dimensional" side, the energy transfer becomes increasingly more difficult and the regime in k space for transfer diminishes. This leads to the fact that part of the spectrum becomes flat as an indication of a thermal equilibrium. We believe the existence of this thermal equilibrium is dictated by the change in the energy transfer and could be a general property in systems where the transfer changes sign. The crossover between the turbulent, cascading part and the thermal equilibrium is determined by balancing the turbulent energy spectrum with a spectrum in thermal equilibrium. We identify the scaling behavior of the crossover point and rescale the spectra accordingly. The PDF of the thermal equilibrium shows simple scaling invariance although the statistics is not Gaussian. For analytical understanding of these results, one rewrites the "GOY" equations in terms of a generating function technique [1,2] thus obtaining a set of coupled ordinary differential equations [20]. One can further map these equations onto a Fokker-Planck equation for the distribution of the exponentiated quantities. We will discuss this in a forthcoming publication.

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[14] This is a forcing dependent feature as it is possible to adjust the forcing to keep both energy and helicity input fixed.
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tistically thermal equilibrium field. From [1] we know that $v_n^2 \approx (\delta_c-\delta)^{-1/3} k_n^{-5/3}$. Since the thermal spectrum is flat $V_n^2 \approx C(\delta)$ one finds the following form of the energy spectrum $E(k_n)=[1/(\delta_c-\delta)]^{1/3} k^{-5/3}+C(\delta)$. Since the turbulent spectrum is infared divergent the main contribution to the energy will be associated to the forcing shell n_f. The equilibrium spectrum is on the other hand dominated by the dissipation scale k_d so the total energy is $E=[1/(\delta_c-\delta)]^{1/3} k_{n_f}^{2/3}+C(\delta) k_d$. As the dissipation scale will vary like $k_d \approx (\delta_c-\delta)^{1/4}$ then $C(\delta) \approx (\delta_c-\delta)^{-7/12}$ which leads to the following expression for the spectrum $E(k_n)=[1/(\delta_c-\delta)]^{1/3} k^{-5/3}+O(\delta_c-\delta)^{-7/12}$, $k_c \approx (\delta_c-\delta)^{3/20}$. For $k>k_c$ the thermal equilibrium spectrum exceeds the turbulent spectrum.

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